

Analogue of Gauss-Lucas theorem for non convex set on the complex plane

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Abstract

Let $S(\phi) = \{z : |\arg(z)| \geq \phi\}$ be a sector on the complex plane \mathcal{C} . If $\phi \geq \pi/2$, then $S(\phi)$ is a convex set and, according to the Gauss-Lucas theorem, if a polynomial $p(z)$ has all its zeros on $S(\phi)$, then the same is true for the zeros of all its derivatives. In this paper is proved that if the polynomial $p(z)$ is with real and non negative coefficients, then the same is true also for $\phi < \pi/2$, when the sector is not a convex set on the complex plane.

Keywords: Gauss-Lucas theorem, polynomials with non negative coefficients.

1 Introduction

We start with defining several notations.

Let R_n be the set of all monic algebraic polynomials with real coefficients and R_n^+ be the subset of R_n containing all polynomials with non negative coefficients. Obviously R_n and R_n^+ are convex sets. Let

$$\begin{aligned} p(z) &= (z - z_1(p))(z - z_2(p)) \cdots (z - z_n(p)) \\ &= z^n + \sum_{k=0}^{n-1} a_k(p)z^k; \quad z_k(p) = r_k(p)e^{i\varphi_k(p)}, \quad k = 1, 2, \dots, n. \end{aligned}$$

For every $t \geq 0$ and $p(z) \in R_n^+$ define the polynomial $p_t(z) \in R_n^+$ by

$$a_0(p_t) = t, \quad a_k(p_t) = a_k(p); \quad k = 1, 2, \dots, n-1.$$

Let $z_k(t) = r_k(t)e^{i\varphi_k(t)}$; $k = 1, 2, \dots, n$ be the zeros of $p_t(z)$. The arguments of the zeros of $p_t(z)$ do not change by the transformation $z \Rightarrow t^{1/n}z$ and $p_t^*(z) = t^{-1}p_t(t^{1/n}z) \in R_n^+$. On the other hand, for $q(z) = z^n + 1$, we have

$$\lim_{t \rightarrow \infty} |p_t^*(z) - q(z)| = 0,$$

hence, according to the Hurwitz theorem,

$$\lim_{t \rightarrow \infty} \varphi_k(p_t^*) = \frac{(2k-1)\pi}{n}; \quad k = 1, 2, \dots, [n/2]. \quad (1)$$

We index the zeros of $p(t)$ on the upper half plane by order of increasing argument of the limit of the zeros of $p_t(z)$ for $t \rightarrow \infty$, or

$$\lim_{t \rightarrow \infty} \varphi_1(p_t) \leq \lim_{t \rightarrow \infty} \varphi_2(p_t) \leq \cdots \leq \lim_{t \rightarrow \infty} \varphi_{[n/2]}(p_t) \leq \pi. \quad (2)$$

Denote by $S(\phi)$ the sector on the complex plane \mathcal{C} ,

$$S(\phi) = \{z : |\arg(z)| \geq \phi\} \quad (3)$$

and by $R_n(\phi)$ ($R_n^+(\phi)$), the set of all polynomials $p(z) \in R_n$ ($p(z) \in R_n^+$), with zeros on the sector (3).

Define the function

$$\varphi_p(t) = \min\{\varphi_1(p_t), \varphi_2(p_t), \dots, \varphi_{[n/2]}(p_t)\}; \quad t \geq 0 \quad \text{and} \quad \varphi(p) = \varphi_p(a_0(p)). \quad (4)$$

If $\varphi(p) \geq \pi/2$, then the sector $S(\varphi(p))$ is a convex set and according to the Gauss-Lucas theorem, we have:

Statement 1 *If $p(z) \in R_n$; $n \geq 2$ and $\varphi(p) \geq \pi/2$, then all the zeros of the derivative $p'(z)$ are on $S(\varphi(p))$.*

Our goal is to show that in Statement 1, the condition $\varphi(p) \geq \pi/2$ is redundant, if $p(z) \in R_n^+$, and to prove:

Theorem 1 *If $p(z) \in R_n^+$; $n \geq 2$ and all the zeros of $p(z)$ are on the sector (3), then all the zeros of its derivative $p'(z)$ are also on the sector (3).*

For $\phi < \pi/2$, the sector (3) is not a convex set on the complex plane, so Theorem 1 is an analogue of the Gauss-Lucas theorem for non convex set. Observe that the differentiation do not change sine of the coefficients, hence:

Corollary 1 *If $p(z) \in R_n^+$; $n \geq 2$ and all the zeros of $p(z)$ are on the sector (3), then all the zeros of all its derivatives $p^{(k)}(z)$; $k = 1, 2, \dots, n-1$ are also on the sector (3).*

2 Preliminaries

It is obvious that a polynomial $p(z) \in R_n^+$ has no real positive zeros, hence $\varphi(p) > 0$ and, more precisely:

Statement 2 *If $p(z) \in R_k^+$, then $\varphi(p) \geq \pi/k$; $k = 1, 2, 3, \dots$*

Really, $\text{Im}(p(z_1(p))) = \sum_{s=1}^k a_s r(p)^s \sin(s\varphi(p)) > 0$, if $\varphi(p) < \pi/k$.

Statement 3 *The set $R_2(\phi) = \{z^2 - 2vrz + r^2 : v \in [-1, \cos \phi], r \geq 0\}$; $\phi \in (0, \pi)$ is convex.*

Proof. As R_2 is a half strip, it is sufficient to prove the convexity separately in respect to v and in respect to r .

Let $v < w$; $v, w \in [-1, \cos \phi]$ and consider the polynomial

$$h(z) = \lambda(z^2 - 2vrz + r^2) + (1 - \lambda)(z^2 - 2wrz + r^2); \quad \lambda \in [0, 1].$$

We have

$$h(z) = z^2 - 2[\lambda v + (1 - \lambda)w]r + r^2 = (z - z_1)(z - \bar{z}_1).$$

Denote $u = \cos \varphi = \lambda v + (1 - \lambda)w \in [v, w]$. Then $z_1 = r(\cos \varphi + i \sin \varphi)$ and convexity in respect to v is proved.

Let $r_1 < r_2$ and consider the function

$$\begin{aligned} h(z) &= \lambda(z^2 - 2vr_1z + r_1^2) + (1 - \lambda)(z^2 - 2vr_2z + r_2^2) \\ &= z^2 - 2v[\lambda r_1 + (1 - \lambda)r_2]z + \lambda r_1^2 + (1 - \lambda)r_2^2 = (z - z_1)(z - \bar{z}_1), \end{aligned}$$

where

$$z_1 = v[\lambda r_1 + (1 - \lambda)r_2] + i\sqrt{\lambda r_1^2 + (1 - \lambda)r_2^2 - v^2[\lambda r_1 + (1 - \lambda)r_2]^2}.$$

As $v \in (0, 1)$, we have

$$\begin{aligned} & \lambda r_1^2 + (1 - \lambda)r_2^2 - v[\lambda r_1 + (1 - \lambda)r_2]^2 \\ & \geq \lambda r_1^2 + (1 - \lambda)r_2^2 - [\lambda r_1 + (1 - \lambda)r_2]^2 = \lambda(1 - \lambda)(r_1 - r_2)^2 > 0. \end{aligned}$$

Hence

$$\tau = \frac{\lambda r_1^2 + (1 - \lambda)r_2^2}{v^2[\lambda r_1 + (1 - \lambda)r_2]^2} > 1$$

and

$$|z_1|^2 = \left| v[\lambda r_1 + (1 - \lambda)r_2] + i\sqrt{\lambda r_1^2 + (1 - \lambda)r_2^2 - v^2[\lambda r_1 + (1 - \lambda)r_2]^2} \right|^2 = \lambda r_1^2 + (1 - \lambda)r_2^2.$$

We prove that $r_1 < |z_1| < r_2$. On the other hand, we have

$$z_1 = [\lambda r_1 + (1 - \lambda)r_2] \cos \phi + i\sqrt{\tau - \cos^2 \phi} = [\lambda r_1 + (1 - \lambda)r_2] \left(\cos \phi + i\sqrt{\sin^2 \phi + \tau - 1} \right).$$

As $\tau > 1$, from the last equality follows that $\arg(z_1) > \phi$, which completes the proof.

Corollary 2 *The set*

$$A_n(g; \phi) = \left\{ p(z) : p(z) = u(z)g(z), u(z) \in R_2(\phi), g(z) \in R_{n-2}(\phi) \right\}$$

is a convex subset of $R_n(\phi)$.

The proof follows from the convexity of $R_2(\phi)$.

Statement 4 *The set of polynomials*

$$B_n(g, g^*; \phi) = \left\{ h(z) : h(z) = \lambda p(z) + (1 - \lambda)f(z), p(z) \in A_n(g; \phi), f(z) \in A_n(g^*; \phi) \right\}, \lambda \in [0, 1]$$

is a convex subset of $R_n(\phi)$.

Proof. For $n = 2$, we have $B_n(g, g^*; \phi) = R_2$, hence the Statement is true, according to Statement 3. For $n = 3$, $g(z) = z + c$, $g^*(z) = z + c^*$, $c, c^* \geq 0$ and $\lambda g(z) + (1 - \lambda)g^*(z) \in R_1(\pi) \subset R_1(\phi)$ is convex. Then $B_3(g, g^*; \phi)$ is a Cartesian product of a half strip and a convex set, hence it is convex.

Suppose that $B_{n-2}(g, g^*; \phi)$ is convex and let $p(z) \in R_n(\phi)$. It is clear that we may factored a quadratic polynomial $u(z) = z^2 - 2rz \cos \varphi + z^2$ with $\varphi \in [\phi, \pi]$, such that $p(z) = u(z)g(z)$, where $g(z) \in R_{n-2}(\phi)$. By induction, $B_n(g, g^*; \phi)$ is a Cartesian product of a half strip and a convex set, hence it is convex. This completes the proof.

Lemma 1 *Let $p(z) \in R_n^+$ and $z_k(t) = r_k(t)e^{i\varphi_k(t)}$; $k = 1, 2, \dots, n$ be the zeros of $p_t(z)$. Then, the function $\varphi_k(t)$ is monotone decreasing and the function $r_k(t)$ is monotone increasing; $k = 1, 2, \dots, [n/2]$.*

Proof. The functions $\varphi_k(t) = \alpha(t)$ and $r_k(t) = \rho(t)$ are smooth in $(0, \infty)$. If the Lemma is not true for $\alpha(t)$, then for a given $t^* \in (0, \infty)$, we shall have $\alpha'(t^*) = 0$. As the zeros of a polynomial are continuous functions of its coefficients, by a small local perturbation, we may have

$$\operatorname{Im}(p'_{t^*}(z_1(p_{t^*}))) \neq 0, \quad (5)$$

preserving the condition $\alpha'(t^*) = 0$. Observe that we may suppose that all the coefficients of $p(z)$ are strictly positive, as such polynomials are everywhere dense in R_n^+ . This allows local perturbations of $p(z)$ without going out of R_n^+ .

Differentiating the equation $p_{t^*}(z_1(p_{t^*})) = 0$ in respect to t we get

$$p'_{t^*}(z_1(p_{t^*}))e^{i\alpha(t^*)}\left(\rho'(t^*) + i\rho(t^*)\alpha'(t^*)\right) + 1 = 0. \quad (6)$$

Denote $\mathcal{Re}\left(p'_{t^*}(z_1(p_{t^*}))\right) = A$ and $\mathcal{Im}\left(p'_{t^*}(z_1(p_{t^*}))\right) = B$. Then, from the equation (6) we get the system

$$A\rho'(t^*) - B\rho(t^*)\alpha'(t^*) = -1, \quad B\rho'(t^*) + A\rho(t^*)\alpha'(t^*) = 0 \quad (7)$$

with determinant $\rho(t^*)(A^2 + B^2) > 0$. Solving the system (7), we get

$$\alpha'(t^*) = \frac{B}{\rho(t^*)(A^2 + B^2)} \quad \text{and} \quad \rho'(t^*) = \frac{A}{(A^2 + B^2)}.$$

If $\alpha'(t^*) > 0$, then $B > 0$, which contradicts (5). This completes the proof that $\alpha(t)$ is monotone.

In the same way, we prove that $\rho(t)$ is monotone. From $\lim_{t \rightarrow \infty} \rho(t) = \infty$, follows that $\rho(t)$ is monotone increasing.

It remained to prove that $\alpha(t)$ is decreasing. We know that

$$\lim_{t \rightarrow \infty} \alpha(t) = \frac{(2k-1)\pi}{n}. \quad (8)$$

There exists $R_0 > 0$, such that for $\rho(t) > R_0$, the inequality

$$\frac{(2k-1)\pi}{n} < \alpha(t) < \frac{2k\pi}{n} \quad (9)$$

holds. Really, $\rho(t)$ may go to ∞ if $\sin n\alpha(t) < 0$. This follows from the equation

$$\mathcal{Im}\left(p_t(z_k(t))\right) = \rho(t)^n \sin n\alpha(t) + \dots + a_1 \rho(t) \sin \alpha(t) = 0.$$

On the other hand, from

$$\mathcal{Re}\left(p_t(z_k(t))\right) = \rho(t)^n \cos n\alpha(t) + \dots + a_1 \rho(t) \cos \alpha(t) + t = 0$$

follows that $\rho(t)$ may go to ∞ if $\cos n\alpha(t) < 0$. From $\sin n\alpha(t) < 0$ and $\cos n\alpha(t) < 0$ follows that

$$\frac{(2k-1)\pi}{n} < \alpha(t) < \frac{(4k-1)\pi}{2n}.$$

As $\rho(t)$ is monotone increasing, from the last and (8), follows that when t is increasing, $\alpha(t)$ is decreasing. This completes the proof.

Corollary 3 *For every polynomial $p(z) \in R_n^+$ the function $\varphi(p)$, see (4), is monotone decreasing.*

3 Proof of Theorem 1

The proof of Theorem 1 will be by induction, based on the main theorem in [1, p. 78, Theorem 1.1], which may be stated, using our notations, as follows:

Theorem 2 If $p(z) \in R_n^+(\phi)$ and

$$p(z) = \left(z^2 - 2zr(p) \cos \varphi(p) + r(p)^2 \right) \hat{p}(z),$$

then $\hat{p}(z) \in R_{n-2}^+(\phi)$.

This very important for us theorem says that a conjugate pair of zeros can be factored from every polynomial $p(z) \in R_n^+(\phi)$; $n \geq 3$ so that the resulting polynomial is in $R_{n-2}^+(\phi)$. That gives the opportunity to prove Theorem 1 by induction.

From Corollary 3 follows that it is sufficient to prove Theorem 1 only for polynomials $p(z) = zq(z)$, $q(z) \in R_{n-1}^+(\phi)$.

For $n = 2$, we have to consider the polynomials $p(z) = zq(z) = z(z+c)$; $c \geq 0$. As $p'(z) = 2z+c$, the Theorem 1 is true for $n = 2$.

For $n = 3$, we have to consider the polynomials $p(z) = z(z^2 - 2rz \cos \varphi + r^2) \in R_3^+(\phi)$; $\phi \in [\pi/3, \pi/2]$. Hence $\varphi = \pi/2$, $p(z) = z(z^2 + r^2)$. This proves Theorem 1 for $n = 3$.

Suppose that Theorem 1 is true for all natural numbers which are less or equal to $n-1$. According to Theorem 2, for $p(z) = zq(z) \in R_n^+(\phi)$, we have

$$p(z) = zq(z) = (z^2 - 2vzr + r^2)zg(z), \quad g(z) \in R_{n-3}^+(\phi), \quad v \in [0, \cos \phi].$$

On the other hand

$$n^{-1}p'(z) = n^{-1}[(3z^2 - 2vzr + r^2)g(z) + (z^2 - vzr + z^2)zg'(z)] \in R_{n-1}^+(\phi). \quad (10)$$

We have to prove that $n^{-1}p'(z) \in R_{n-1}^+(\phi)$. From Corollary 2 follows that $(3z^2 - 2vzr + r^2)g(z) \in A(g, \phi)$, as the zeros of $3z^2 - 2vzr + r^2 = 0$ are

$$z_{1,2} = \frac{r}{3} \left(\cos \phi \pm i \sqrt{\sin^2 \phi + 2} \right)$$

and the argument of z_1 is bigger than ϕ

We have also, by induction, that $g^*(z) = zg'(z) \in A(g^*, \phi)$. From Statement 4 follows that $n^{-1}p'(z) \in R_{n-1}(\phi)$ and according to (10) $p'(z) \in R_{n-1}^*(\phi)$. This completes the proof of Theorem 1.

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References

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